THE EXCEPTIONAL JORDAN EIGENVALUE PROBLEM *

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Abstract

We discuss the eigenvalue problem for 3×3 octonionic Hermitian matrices which is relevant to the Jordan formulation of quantum mechanics. In contrast to the eigenvalue problems considered in our previous work, all eigenvalues are real and solve the usual characteristic equation. We give an elementary construction of the corresponding eigenmatrices, and we further speculate on a possible application to particle physics.

1 Introduction

In previous work [1, 2, 3] we considered both the *left* and *right* eigenvalue problems for 2×2 and 3×3 octonionic Hermitian matrices, given explicitly by

$$\mathcal{A}\, v = \lambda\, v \tag{1}$$

and

$$\mathcal{A}\, v = v\, \lambda \tag{2}$$

respectively. We showed in [1] that the left eigenvalue problem admits nonreal eigenvalues over both the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$, while the right eigenvalue problem admits nonreal eigenvalues only over $\mathbb{O}$. Some of the intriguing properties of the eigenvectors corresponding to these nonreal eigenvalues were considered in [3], and in [4, 5] we discussed possible applications to physics, including the remarkable fact that simultaneous eigenvectors of all 3 angular momentum operators exist in this context.

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However, the main result in [1] concerned real eigenvalues in the 3×3 octonionic case. For this case, there are 6, rather than 3, real eigenvalues [6]. We showed that these come in 2 independent families, each consisting of 3 real eigenvalues which satisfy a modified characteristic equation rather than the usual one. Furthermore, the corresponding eigenvectors are not orthogonal in the usual sense, but do satisfy a generalized notion of orthogonality (see also [2, 7]). Finally, all such matrices admit a decomposition in terms of (the "squares" of) orthonormal eigenvectors. However, due to associativity problems, these matrices are *not* idempotents (matrices which square to themselves).

It is the purpose of this paper to describe a related eigenvalue problem for 3×3 Hermitian octonionic matrices which does have the standard properties: There are 3 real eigenvalues, which solve the usual characteristic equation, and which lead to a decomposition in terms of orthogonal "eigenvectors" which are indeed (primitive) idempotents.

This is accomplished by considering the *eigenmatrix* problem

$$\mathcal{A} \circ \mathcal{V} = \lambda \mathcal{V} \tag{3}$$

where $\mathcal{V}$ is itself an octonionic Hermitian matrix and $\circ$ denotes the *Jordan product* [8, 9]

$$\mathcal{A} \circ \mathcal{B} = \frac{1}{2} (\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) \tag{4}$$

which is commutative but not associative. We further restrict $\mathcal{V}$ to be a (primitive) idempotent; as discussed below, this ensures that the Jordan eigenvalue problem (3) reduces to the traditional eigenvalue problem (2) in the non-octonionic cases.

The exceptional Jordan algebra of 3×3 octonionic Hermitian matrices under the Jordan product, now known as the Albert algebra, was extensively studied by Freudenthal [10, 11, 12], [1] and is well-known to mathematicians [13, 14, 15, 16]. In particular, the existence of a decomposition in terms of orthogonal idempotents, and its relationship to the eigenvalue problem (4), was shown already in [9]. Furthermore, since any Jordan matrix can be diagonalized by an F_4 transformation [11], and since F_4 is the automorphism group of the Jordan product [17], the eigenmatrix problem (3) is easily solved in theory. However, we are not aware of an elementary treatment along the lines presented here.

Our motivation for studying this problem is the well-known fact that the Albert algebra is the only exceptional realization of the Jordan formulation of quantum mechanics [8, 9, 18, 19]; over an associative division algebra, the Jordan formalism reduces to standard quantum mechanics. Furthermore, the 4 division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are fundamentally associated with the Killing/Cartan classification of Lie algebras — corresponding to physical symmetry groups — into orthogonal, unitary, symplectic, and exceptional types. This most exceptional quantum mechanical system over the most exceptional division algebra provides an intriguing framework to study the basic symmetries of nature.

We begin by summarizing the properties of the Albert algebra in Section 2. In order to make our work accessible to a wider audience, we first motivate our subsequent computation by briefly reviewing the Jordan formulation of quantum mechanics in Section 3, before

[1] Freudenthal's early work on this topic was originally distributed in German in mimeographed form [10], parts of which were later summarized in [11], which we henceforth cite. Many of these results can also be found in English in [12].

presenting the mathematical details of the eigenvalue results in Section 4. In Section 5, we include a brief but suggestive discussion of possible applications, such as its relevance for our recent work on dimensional reduction [4, 5]. Finally, in the Appendix, we show explicitly how to diagonalize a generic Jordan matrix using F_4 transformations.

2 The Albert Algebra

We consider the *Albert algebra* consisting of 3×3 octonionic Hermitian matrices, which we will call *Jordan matrices.* [2] The Jordan product (4) of two such matrices is commutative but not associative. We have in particular that

$$\mathcal{A}^2 \equiv \mathcal{A} \circ \mathcal{A} \tag{5}$$

and we *define*

$$\mathcal{A}^3 := \mathcal{A}^2 \circ \mathcal{A} \equiv \mathcal{A} \circ \mathcal{A}^2 \tag{6}$$

which differs from the cube of $\mathcal{A}$ using ordinary matrix multiplication. Other operations on Jordan matrices are the *trace*, denoted as usual by $\mathrm{tr}(\mathcal{A})$, and the *Freudenthal product* [11]

$$\mathcal{A} * \mathcal{B} = \mathcal{A} \circ \mathcal{B} - \frac{1}{2}\Big(\mathcal{A}\,\mathrm{tr}(\mathcal{B}) + \mathcal{B}\,\mathrm{tr}(\mathcal{A})\Big) + \frac{1}{2}\Big(\mathrm{tr}(\mathcal{A})\,\mathrm{tr}(\mathcal{B}) - \mathrm{tr}(\mathcal{A} \circ \mathcal{B})\Big)\, I \tag{7}$$

where I denotes the identity matrix and with the important special case

$$\mathcal{A} * \mathcal{A} = \mathcal{A}^2 - (\mathrm{tr}\,\mathcal{A})\,\mathcal{A} + \sigma(\mathcal{A})\, I \tag{8}$$

where

$$\sigma(\mathcal{A}) = \frac{1}{2}\left((\mathrm{tr}\,\mathcal{A})^2 - \mathrm{tr}(\mathcal{A}^2)\right) \equiv \mathrm{tr}(\mathcal{A} * \mathcal{A}) \tag{9}$$

There is also *trace reversal*

$$\widetilde{\mathcal{A}} = \mathcal{A} - \mathrm{tr}(\mathcal{A})\, I \equiv -2\, I * \mathcal{A} \tag{10}$$

and, finally, the *determinant*

$$\det(\mathcal{A}) = \frac{1}{3}\,\mathrm{tr}\Big((\mathcal{A} * \mathcal{A}) \circ \mathcal{A}\Big) \tag{11}$$

which can equivalently be defined by

$$\Big((\mathcal{A} * \mathcal{A}) \circ \mathcal{A}\Big) = (\det \mathcal{A})\, I \tag{12}$$

Expanding (12) using (8), we obtain the remarkable result that Jordan matrices satisfy the usual characteristic equation [11]

$$\mathcal{A}^3 - (\mathrm{tr}\,\mathcal{A})\,\mathcal{A}^2 + \sigma(\mathcal{A})\,\mathcal{A} - (\det \mathcal{A})\, I = 0 \tag{13}$$

[2] For a review of the basic properties of the octonions, see for instance [1] or [19, 20].

Explicitly, a Jordan matrix can be written as

$$\mathcal{A} = \begin{pmatrix} p & a & \overline{b} \\ \overline{a} & m & c \\ b & \overline{c} & n \end{pmatrix} \tag{14}$$

with $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$, where the bar denotes octonionic conjugation. The definitions above then take the concrete form

$$\begin{aligned} \operatorname{tr} \mathcal{A} &= p + m + n \\ \sigma(\mathcal{A}) &= pm + mn + pn - |a|^2 - |b|^2 - |c|^2 \\ \det \mathcal{A} &= pmn + b(ac) + \overline{b(ac)} - n|a|^2 - m|b|^2 - p|c|^2 \end{aligned} \tag{15}$$

The *Cayley plane*, also called the *Moufang plane*, consists of those Jordan matrices $\mathcal{V}$ which satisfy the restriction [12, 15]

$$\mathcal{V} \circ \mathcal{V} = \mathcal{V}; \qquad \operatorname{tr} \mathcal{V} = 1 \tag{16}$$

We will see below that elements of the Cayley plane correspond to projection operators in the Jordan formulation of quantum mechanics. As shown in [15], the conditions (16) force the components of $\mathcal{V}$ to lie in a *quaternionic* subalgebra of $\mathbb{O}$ (which depends on $\mathcal{V}$). Basic (associative) linear algebra then shows that each element of the Cayley plane is a primitive idempotent (an idempotent which is *not* the sum of other idempotents), and can be written as

$$\mathcal{V} = vv^\dagger \tag{17}$$

where v is a 3-component octonionic column vector, whose components lie in the quaternionic subalgebra determined by $\mathcal{V}$, and which is normalized by

$$v^\dagger v = \operatorname{tr} \mathcal{V} = 1 \tag{18}$$

Note that v is unique up to a *quaternionic* phase. Furthermore, using (8) and its trace (9), it is straightforward to show that, for any Jordan matrix $\mathcal{B}$,

$$\mathcal{B} * \mathcal{B} = 0 \Longleftrightarrow \mathcal{B} \circ \mathcal{B} = (\operatorname{tr} \mathcal{B})\, \mathcal{B} \tag{19}$$

which agrees with (16) up to normalization, and which is therefore the condition that that $\pm\mathcal{B}$ can be written in the form (17) (without the restriction (18)). Note further that for any Jordan matrix satisfying (19), the normalization $\operatorname{tr} \mathcal{B}$ can only be zero if v, and hence $\mathcal{B}$ itself, is zero, so that

$$\mathcal{B} * \mathcal{B} = 0 = \operatorname{tr} \mathcal{B} \Longleftrightarrow \mathcal{B} = 0 \tag{20}$$

since the converse is obvious.

We will need the following useful identities

$$(\mathcal{A} * \mathcal{A}) * (\mathcal{A} * \mathcal{A}) = (\det \mathcal{A})\, \mathcal{A} \tag{21}$$

$$(\widetilde{\mathcal{A}} \circ \mathcal{A}) \circ (\mathcal{A} * \mathcal{A}) = (\det \mathcal{A})\, \widetilde{\mathcal{A}} \tag{22}$$

for any Jordan matrix $\mathcal{A}$, which can be verified by direct computation. Finally, we also have the remarkable fact that

$$\mathcal{A} * \mathcal{A} = 0 = \mathcal{B} * \mathcal{B} \Longrightarrow (\mathcal{A} * \mathcal{B}) * (\mathcal{A} * \mathcal{B}) = 0 \tag{23}$$

which follows by polarizing (21) [3] and which ensures that the set of Jordan matrices satisfying (19), consisting of all real multiples of elements of the Cayley plane, is closed under the Freudenthal product.

Before proceeding further it is illuminating to consider the restriction to *real* column vectors. If $u, v, w \in \mathbb{R}^3$, then

$$\begin{aligned} 2\, uu^\dagger \circ vv^\dagger &= (u \cdot v)\,(uv^\dagger + vu^\dagger) && (24) \\ \operatorname{tr}(uu^\dagger \circ vv^\dagger) &= (u \cdot v)^2 && (25) \end{aligned}$$

where $\cdot$ denotes the usual dot product (and where the Hermitian conjugate of a real matrix is of course just its transpose). We also have

$$2\, uu^\dagger * vv^\dagger = (u \times v)(u \times v)^\dagger \tag{26}$$

where $\times$ denotes the usual cross product. We can therefore view the Jordan product as a generalization of the (square of the) dot product, and the Freudenthal product as a generalization of the (square of the) cross product.

This somewhat simplified perspective is nevertheless extremely useful in grasping the essential content of the corresponding octonionic manipulations. For instance, the linear independence of (real) u, v, w is given by the condition

$$\det(Q) = u \cdot (v \times w) \neq 0 \tag{27}$$

where Q is the matrix whose columns are the vectors u, v, w. Note that

$$QQ^\dagger \equiv uu^\dagger + vv^\dagger + ww^\dagger \tag{28}$$

and of course $\det(QQ^\dagger) = |\det(Q)|^2$. But using the definition (11) for real u, v, w leads to the identity

$$\det(uu^\dagger + vv^\dagger + ww^\dagger) = \Big(u \cdot (v \times w)\Big)^2 \tag{29}$$

which not only emphasizes the role played by the determinant in determining linear independence, but also makes plausible the cyclic nature of the trace of the triple product obtained by polarizing (11).

3 The Jordan Formulation of Quantum Mechanics

In the Dirac formulation of quantum mechanics, a quantum mechanical state is represented by a *complex* vector v, often written as $|v\rangle$, which is usually normalized such that $v^\dagger v = 1$.

[3] The necessary fact that $\det(\mathcal{A} + \mathcal{B}) = 0$ follows from the definition (11) of the determinant in terms of the triple product, the cyclic properties of the trace of the triple product, and the assumptions on $\mathcal{A}$ and $\mathcal{B}$.

In the Jordan formulation [8, 9, 19], the same state is instead represented by the Hermitian matrix $vv^\dagger$, also written as $|v\rangle\langle v|$, which squares to itself and has trace 1 (compare (16)). The matrix $vv^\dagger$ is thus the projection operator for the state v, which can also be viewed as a pure state in the density matrix formulation of quantum mechanics. Note that the phase freedom in v is no longer present in $vv^\dagger$, which is uniquely determined by the state (and the normalization condition).

A fundamental object in the Dirac formalism is the probability amplitude $v^\dagger w$, or $\langle v|w\rangle$, which is not however measurable; it is the the squared norm $|\langle v|w\rangle|^2 = \langle v|w\rangle\langle w|v\rangle$ of the probability amplitude which yields the measurable transition probabilities. One of the basic observations which leads to the Jordan formalism is that these transition probabilities can be expressed entirely in terms of the Jordan product of projection operators, since

$$(v^\dagger w)(w^\dagger v) \equiv \mathrm{tr}(vv^\dagger \circ ww^\dagger) \tag{30}$$

A similar but less obvious translation scheme also exists [19] for transition probabilities of the form $|\langle v|A|w\rangle|^2$, where A is a Hermitian matrix, corresponding (in both formalisms) to an observable, so that all *measurable* quantities in the Dirac formalism can be expressed in the Jordan formalism.

So far, we have assumed that the state vector v and the observable A are complex. But the Jordan formulation of quantum mechanics uses only the *Jordan identity*

$$(A \circ B) \circ A^2 = A \circ \left(B \circ A^2\right) \tag{31}$$

for 2 observables (Hermitian matrices) A and B. As shown in [9], the Jordan identity (31) is equivalent to power associativity, which ensures that arbitrary powers of Jordan matrices — and hence of quantum mechanical observables — are well-defined.

The Jordan identity (31) is the defining property of a *Jordan algebra* [8], and is clearly satisfied if the operator algebra is associative, which will be the case if the elements of the Hermitian matrices A, B themselves lie in an associative algebra. Remarkably, the only further possibility is the Albert algebra of 3×3 octonionic Hermitian matrices [9, 18]. [4] In what follows we will restrict our attention to this exceptional case.

4 The Jordan Eigenvalue Problem

Consider finally the eigenmatrix problem (3). Note first of all that since $\mathcal{A}$ and $\mathcal{V}$ are Jordan matrices, the left-hand-side is Hermitian, which forces λ to be real.

Suppose first that $\mathcal{A}$ is diagonal. Then the diagonal elements p, m, n are clearly eigenvalues, with obvious diagonal eigenmatrices. But there are also other "eigenvalues", namely the averages $(p+m)/2$, $(m+n)/2$, $(n+p)/2$. However, the corresponding eigenmatrices — which are related to Peirce decompositions [13, 14] — have only zeros on the diagonal. Thus, by (20), they can not satisfy (16), and hence can not be written in the form (17). To exclude this case, we therefore restrict $\mathcal{V}$ in (3) to the Cayley plane (16), which ensures that

[4] The 2×2 octonionic Hermitian matrices also form a Jordan algebra, but, even though the octonions are not associative, it is possible to find an associative algebra which leads to the same Jordan algebra [9, 13].

the eigenmatrices $\mathcal{V}$ are primitive idempotents; they really do correspond to "eigenvectors" v. Recall that this forces the components of $\mathcal{V}$ to lie in a quaternionic subalgebra of $\mathbb{O}$ (which depends on $\mathcal{V}$) even though the components of $\mathcal{A}$ may not.

Next consider the characteristic equation

$$-\det(\mathcal{A}-\lambda I) = \lambda^3 - (\operatorname{tr}\mathcal{A})\,\lambda^2 + \sigma(\mathcal{A})\,\lambda - (\det\mathcal{A})\,I = 0 \tag{32}$$

It is not at first obvious that all solutions λ of (32) are real. To see that this is indeed the case, we note that $\mathcal{A}$ can be rewritten as a 24×24 real symmetric matrix, whose eigenvalues are of course real. However, as discussed in [1], these latter eigenvalues do *not* satisfy the characteristic equation (32)! Rather, they satisfy a modified characteristic equation of the form

$$\det(\mathcal{A}-\lambda I) + r = 0 \tag{33}$$

where r is either of the roots of a quadratic equation which depends on $\mathcal{A}$. As shown explicitly using *Mathematica* in Figure 5 of [2], not only are these roots real, but they have opposite signs (or at least one is zero). But, as can be seen immediately using elementary graphing techniques, if the cubic equation (33) has 3 real roots for both a positive and a negative value of r, it also has 3 real roots for all values of r in between, including $r = 0$. This shows that (32) does indeed have 3 real roots.

Alternatively, since F_4 preserves both the determinant and the trace (and therefore also σ) [11, 15], it leaves the characteristic equation invariant. Since F_4 can be used to diagonalize $\mathcal{A}$ [11, 15], and since the resulting diagonal elements clearly satisfy the characteristic equation, we have another, indirect, proof that the characteristic equation has 3 real roots. Furthermore, this shows that these roots correspond precisely to the 3 real eigenvalues whose eigenmatrices lie in the Cayley plane. We therefore reserve the word "eigenvalue" for the 3 solutions of the characteristic equation (32), explicitly excluding their averages. The above argument shows that these correspond to solutions $\mathcal{V}$ of (3) which lie in the Cayley plane; we will verify this explicitly below.

Restricting the eigenvalues in this way corresponds to the traditional eigenvalue problem in the following sense. If $\mathcal{A}$, $v \neq 0$ lie in a quaternionic subalgebra of the octonions, then the Jordan eigenvalue problem (3) together with the restriction (16) becomes

$$\mathcal{A}\,vv^\dagger + vv^\dagger\mathcal{A} = 2\lambda\, vv^\dagger \tag{34}$$

Multiplying (34) on the right by v and simplifying the result using the trace of (34) leads immediately to $Av = \lambda v$ (with $\lambda \in \mathbb{R}$), that is, the Jordan eigenvalue equation implies the ordinary eigenvalue equation in this context. Since the converse is immediate, the Jordan eigenvalue problem (3) (with $\mathcal{V}$ restricted to the Cayley plane but $\mathcal{A}$ octonionic) is seen to be a reasonable generalization of the ordinary eigenvalue problem.

We now show how to construct eigenmatrices $\mathcal{V}$ of (3), restricted to lie in the Cayley plane, and with real eigenvalues λ satisfying the characteristic equation (32). From the definition of the determinant, we have for real λ satisfying (32)

$$0 = \det(\mathcal{A}-\lambda I) = (\mathcal{A}-\lambda I)\circ\Big((\mathcal{A}-\lambda I)*(\mathcal{A}-\lambda I)\Big) \tag{35}$$

Thus, setting

$$\mathcal{Q}_\lambda = (\mathcal{A} - \lambda\, I) * (\mathcal{A} - \lambda\, I) \tag{36}$$

we have

$$(\mathcal{A} - \lambda\, I) \circ \mathcal{Q}_\lambda = 0 \tag{37}$$

so that $\mathcal{Q}_\lambda$ is a solution of (3).

Due to the identity (21), we have

$$\mathcal{Q}_\lambda * \mathcal{Q}_\lambda = 0 \tag{38}$$

If $\mathcal{Q}_\lambda \neq 0$, we can renormalize $\mathcal{Q}_\lambda$ by defining

$$\mathcal{P}_\lambda = \frac{\mathcal{Q}_\lambda}{\operatorname{tr}(\mathcal{Q}_\lambda)} \tag{39}$$

Each resulting $\mathcal{P}_\lambda$ is in the Cayley plane, and is hence a primitive idempotent. Due to (38), we can write

$$\mathcal{P}_\lambda = v_\lambda v_\lambda^\dagger \tag{40}$$

and we call v_λ the (generalized) eigenvector of $\mathcal{A}$ with eigenvalue λ. Note that v_λ does *not* in general satisfy either (1) or (2). Rather, we have

$$\mathcal{A} \circ v_\lambda v_\lambda^\dagger = \lambda\, v_\lambda v_\lambda^\dagger \tag{41}$$

as well as

$$v_\lambda^\dagger v_\lambda = 1 \tag{42}$$

Writing out all the terms and using (10) and (22), one computes directly that

$$\mathcal{Q}_\lambda \circ (\mathcal{A} \circ \mathcal{Q}_\mu) = (\mathcal{Q}_\lambda \circ \mathcal{A}) \circ \mathcal{Q}_\mu \tag{43}$$

If λ, μ are solutions of the characteristic equation (32), then using (37) leads to

$$\mu\, (\mathcal{Q}_\lambda \circ \mathcal{Q}\mu) = \lambda\, (\mathcal{Q}_\lambda \circ \mathcal{Q}\mu) \tag{44}$$

If we now assume $\lambda \neq \mu$ and $\mathcal{Q}_\lambda \neq 0 \neq \mathcal{Q}_\mu$, this shows that eigenmatrices corresponding to different eigenvalues are orthogonal in the sense

$$\mathcal{P}_\lambda \circ \mathcal{P}_\mu = 0 \tag{45}$$

where we have normalized the eigenmatrices.

We now turn to the case $\mathcal{Q}_\lambda = 0$. We have first that

$$\operatorname{tr}(\mathcal{Q}_\lambda) = \operatorname{tr}\Big((\mathcal{A} - \lambda\, I) * (\mathcal{A} - \lambda\, I)\Big) = \sigma(\mathcal{A} - \lambda\, I) \tag{46}$$

Denoting the 3 real solutions of the characteristic equation (32) by λ, μ, ν, so that

$$\begin{aligned} \operatorname{tr}\mathcal{A} &= \lambda + \mu + \nu && (47)\\ \sigma(\mathcal{A}) &= \lambda(\mu + \nu) + \mu\nu && (48) \end{aligned}$$

we then have

$$\sigma(\mathcal{A} - \lambda I) = \sigma(\mathcal{A}) - 2\lambda \operatorname{tr} \mathcal{A} + 3\lambda^2 = (\lambda - \mu)(\lambda - \nu) \tag{49}$$

But by (38) and (20), $\mathcal{Q}_\lambda = 0$ if and only if $\operatorname{tr}(\mathcal{Q}_\lambda) = 0$. Using (46) and (49), we therefore see that $\mathcal{Q}_\lambda = 0$ if and only if λ is a solution of (32) of multiplicity greater than 1. We will return to this case below.

Putting this all together, if there are no repeated solutions of the characteristic equation (32), then the eigenmatrix problem leads to the decomposition

$$\mathcal{A} = \sum_{i=1}^{3} \lambda_i \mathcal{P}_{\lambda_i} \tag{50}$$

in terms of orthogonal primitive idempotents, which expresses each Jordan matrix $\mathcal{A}$ as a sum of squares of *quaternionic* columns. [5] We emphasize that the components of the eigenmatrices $\mathcal{P}_{\lambda_i}$ need not lie in the same quaternionic subalgebra, and that $\mathcal{A}$ is octonionic. Nonetheless, it is remarkable that $\mathcal{A}$ admits a decomposition in terms of matrices which are, individually, quaternionic.

We now return to the case $\mathcal{Q}_\lambda = 0$, corresponding to repeated eigenvalues. If λ is a solution of the characteristic equation (32) of multiplicity 3, then $\operatorname{tr} \mathcal{A} = 3\lambda$ and $\sigma(\mathcal{A}) = 3\lambda^2$. As shown in [1] in a different context, or using an argument along the lines of Footnote 5, this forces $\mathcal{A} = \lambda I$, which has a trivial decomposition into orthonormal primitive idempotents. We are left with the case of multiplicity 2, corresponding to $\mathcal{A} \neq \lambda I$ and $\mathcal{Q}_\lambda = 0$.

Since $\mathcal{Q}_\lambda = 0$, $\mathcal{A} - \lambda I$ is (up to normalization) in the Cayley plane, and we have

$$\mathcal{A} - \lambda I = \pm w w^\dagger \tag{51}$$

with the components of w in some quaternionic subalgebra of $\mathbb{O}$. While $ww^\dagger$ is indeed an eigenmatrix of $\mathcal{A}$, it has eigenvalue $\mu = \operatorname{tr}(\mathcal{A}) - 2\lambda \neq \lambda$. However, it is straightforward to construct a vector v orthogonal to w in a suitable sense. For instance, if

$$w = \begin{pmatrix} x \\ y \\ r \end{pmatrix} \tag{52}$$

with $r \in \mathbb{R}$, then choosing

$$v = \begin{pmatrix} |y|^2 \\ -y\overline{x} \\ 0 \end{pmatrix} \tag{53}$$

leads to

$$vv^\dagger \circ ww^\dagger = 0 \tag{54}$$

and only minor modifications are required to adapt this example to the general case. But (51) now implies that

$$\mathcal{A} \circ vv^\dagger = \lambda\, vv^\dagger \tag{55}$$

[5] To see this, one easily verifies that $\operatorname{tr}(\mathcal{B}) = 0 = \sigma(\mathcal{B})$, where $\mathcal{B} = \mathcal{A} - \sum \lambda_i \mathcal{P}_{\lambda_i}$. But this implies that $\operatorname{tr}(\mathcal{B}^2) = 0$, which forces $\mathcal{B} = 0$.

so that we have constructed an eigenmatrix of $\mathcal{A}$ with eigenvalue λ.

We can now perturb $\mathcal{A}$ slightly by adding $\epsilon\, vv^\dagger$, thus changing the eigenvalue of $vv^\dagger$ by ϵ. The resulting matrix will have 3 unequal eigenvalues, and hence admit a decomposition (50) in terms of orthogonal primitive idempotents. But these idempotents will also be eigenmatrices of $\mathcal{A}$, and hence yield an orthogonal primitive idempotent decomposition of $\mathcal{A}$. [6] In summary, decompositions analogous to (50) can also be found when there is a repeated eigenvalue, but the terms corresponding to the repeated eigenvalue can not be written in terms of the projections $\mathcal{P}_\lambda$, and of course the decomposition of the corresponding eigenspace is not unique. [7]

5 Discussion

We have argued elsewhere [4, 5] that the ordinary momentum-space (massless and massive) Dirac equation in $3+1$ dimensions can be obtained via dimensional reduction from the Weyl (massless Dirac) equation in $9+1$ dimensions. This latter equation can be written as the eigenvalue problem

$$\widetilde{P}\psi = 0 \tag{59}$$

where P is a 2×2 octonionic Hermitian matrix corresponding to the 10-dimensional momentum and tilde again denotes trace reversal. The general solution of this equation is

$$\begin{aligned} P &= \pm\theta\theta^\dagger && (60)\\ \psi &= \theta\xi && (61) \end{aligned}$$

where θ is a 2-component octonionic vector whose components lie in the same complex subalgebra of $\mathbb{O}$ as do those of P, and where $\xi \in \mathbb{O}$ is arbitrary. (Such a θ must exist since $\det(P) = 0$.)

It is then natural to introduce a 3-component formalism; this approach was used by Schray [21, 22] for the superparticle. Defining

$$\Psi = \begin{pmatrix} \theta \\ \overline{\xi} \end{pmatrix} \tag{62}$$

[6] More formally, with the above assumptions we have

$$(\mathcal{A} + \epsilon\, vv^\dagger - \lambda\, I) * (\mathcal{A} + \epsilon\, vv^\dagger - \lambda\, I) = (ww^\dagger + \epsilon\, vv^\dagger) * (ww^\dagger + \epsilon\, vv^\dagger) = 2\epsilon\, vv^\dagger * ww^\dagger \tag{56}$$

The Freudenthal square of (56) is zero by (23), which shows that $\det(\mathcal{A} + \epsilon\, vv^\dagger - \lambda\, I) = 0$ by (21), so that λ is indeed an eigenvalue of the perturbed matrix $\mathcal{A} + \epsilon\, vv^\dagger$. Furthermore, (56) itself is not zero (unless v or w vanishes) since (54) implies that

$$2\,\mathrm{tr}(vv^\dagger * ww^\dagger) = (v^\dagger v)(w^\dagger w) \neq 0 \tag{57}$$

which shows that λ does not have multiplicity 2.

[7] An invariant orthogonal idempotent decomposition when λ is an eigenvalue of multiplicity 2 is

$$\mathcal{A} = \mu\,\frac{(\mathcal{A} - \lambda\, I)}{\mathrm{tr}(\mathcal{A} - \lambda\, I)} - \lambda\,\frac{(\widetilde{\mathcal{A} - \lambda\, I})}{\mathrm{tr}(\mathcal{A} - \lambda\, I)} \tag{58}$$

where the coefficient of $\mu = \mathrm{tr}(\mathcal{A}) - 2\lambda$ is the primitive idempotent corresponding to the other eigenvalue and the coefficient of λ is an idempotent but not primitive. An equivalent expression was given in [9].

we have first of all that

$$\mathcal{P} := \Psi\Psi^\dagger = \begin{pmatrix} P & \psi \\ \psi^\dagger & |\xi|^2 \end{pmatrix} \tag{63}$$

so that Ψ combines the bosonic and fermionic degrees of freedom. Lorentz transformations can be constructed by iterating ("nesting") transformations of the form [23]

$$P \mapsto MPM^\dagger \tag{64}$$
$$\psi \mapsto M\psi \tag{65}$$

which can be elegantly combined into the transformation

$$\mathcal{P} \mapsto \mathcal{M}\mathcal{P}\mathcal{M}^\dagger \tag{66}$$

with

$$\mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \tag{67}$$

This in fact shows how to view $SO(9,1)$ as a subgroup of E_6; the rotation subgroup $SO(9)$ lies in F_4. It turns out that the Dirac equation (59) is equivalent to the equation

$$\mathcal{P} * \mathcal{P} = 0 \tag{68}$$

which shows both that solutions of the Dirac equation correspond to the Cayley plane and that the Dirac equation admits E_6 as a symmetry group. Using the particle interpretation from [4, 5] then leads to the interpretation of (part of) the Cayley plane as representing 3 generations of leptons.

The modern description of symmetries in nature is in terms of Lie algebras. For instance, one describes angular momentum by taking an infinitesimal rotation, regarding it as a self-adjoint operator, and studying the resulting eigenvalue problem. Thus, if A is the (self-adjoint version of the) infinitesimal rotation M, then the rotation (65) leads to the eigenvalue problem $A\psi = \lambda\psi$. But the infinitesimal form of (64) is essentially $A \circ P$, although in the octonionic case, it is not clear how best to make A self-adjoint. It thus seems natural to study the (3×3) Jordan eigenvalue problem associated with (66).

Finally, we refer to decompositions of the form (50) as p-square decompositions, where p is the number of nonzero eigenvalues, and hence the number of nonzero primitive idempotents in the decomposition. If $\det(\mathcal{A}) \neq 0$, then $\mathcal{A}$ is a 3-square. If $\det(\mathcal{A}) = 0 \neq \sigma(\mathcal{A})$, then $\mathcal{A}$ is a 2-square. Finally, if $\det(\mathcal{A}) = 0 = \sigma(\mathcal{A})$, then $\mathcal{A}$ is a 1-square (unless also $\mathrm{tr}(\mathcal{A}) = 0$, in which case $\mathcal{A} \equiv 0$). It is intriguing that, since E_6 preserves both the determinant and the condition $\sigma(\mathcal{A}) = 0$, E_6 therefore preserves the class of p-squares for each p. If, as argued above, 1-squares correspond to leptons, is it possible that 2-squares are mesons and 3-squares are baryons?

APPENDIX: Diagonalizing Jordan Matrices Using F_4

We start with a Jordan matrix in the form (14), and show how to diagonalize it using nested F_4 transformations. As discussed in [15], a set of generators for F_4 can be obtained by considering its $SO(9)$ subgroups, which in turn can be generated by 2×2 tracefree, Hermitian, octonionic matrices.

Just as for the traditional diagonalization procedure, it is first necessary to solve the characteristic equation for the eigenvalues. Let λ be a solution of (32), and let $vv^\dagger \neq 0$ be a solution of (3) with eigenvalue λ. [8] We assume further that the phase in v is chosen such that

$$v = \begin{pmatrix} x \\ y \\ r \end{pmatrix} \tag{69}$$

where $x, y \in \mathbb{O}$ and $r \in \mathbb{R}$. Define

$$\mathcal{M}_1 = \begin{pmatrix} -r & 0 & x \\ 0 & N_1 & 0 \\ \overline{x} & 0 & r \end{pmatrix} \Big/ N_1 \qquad \mathcal{M}_2 = \begin{pmatrix} N_2 & 0 & 0 \\ 0 & -N_1 & y \\ 0 & \overline{y} & N_1 \end{pmatrix} \Big/ N_2 \tag{70}$$

where the normalization constants are given by $N_1^2 = |x|^2 + r^2$ and $N_2^2 = N_1^2 + |y|^2 \equiv v^\dagger v \neq 0$. (If $N_1 = 0$, then $\mathcal{A}$ is already block diagonal.) It is straightforward to check that

$$\mathcal{M}_2 \mathcal{M}_1 v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{71}$$

and, since everything so far is quaternionic, that

$$\mathcal{M}_2 \mathcal{M}_1 v v^\dagger \mathcal{M}_1 \mathcal{M}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: \mathcal{E}_3 \tag{72}$$

But conjugation by each of the $\mathcal{M}_i$ is an F_4 transformation (which is well-defined since each $\mathcal{M}_i$ separately has components which lie in a *complex* subalgebra of $\mathbb{O}$); this is precisely the form of the generators referred to earlier. Furthermore, F_4 is the automorphism group of the Jordan product (4). Thus, since

$$(\mathcal{A} - \lambda\, vv^\dagger) \circ vv^\dagger = 0 \tag{73}$$

then after applying the (nested!) F_4 transformation above, we obtain

$$\Big(\mathcal{M}_2 \Big(\mathcal{M}_1 (\mathcal{A} - \lambda\, I) \mathcal{M}_1\Big) \mathcal{M}_2\Big) \circ \mathcal{E}_3 = 0 \tag{74}$$

[8] It is straightforward to construct v using the results of Section 4, especially since we can assume without loss of generality that λ is an eigenvalue of multiplicity 1.

which in turn forces

$$\mathcal{M}_2(\mathcal{M}_1\mathcal{A}\mathcal{M}_1)\mathcal{M}_2 = \begin{pmatrix} X & 0 \\ 0 & \lambda \end{pmatrix} \tag{75}$$

where

$$X = \begin{pmatrix} s & z \\ \overline{z} & t \end{pmatrix} \tag{76}$$

is a 2×2 octonionic Hermitian matrix (with $z \in \mathbb{O}$ and $s, t \in \mathbb{R}$).

The final step amounts to the diagonalization of X, which is easy. Let μ be any eigenvalue of X (which in fact means that it is another solution of (32)) and set

$$\mathcal{M}_3 = \begin{pmatrix} \mu - t & 0 & 0 \\ 0 & t - \mu & z \\ 0 & \overline{z} & N_3 \end{pmatrix} \Big/ N_3 \tag{77}$$

where $N_3 = (\mu - t)^2 + |z|^2$. (If $N_3 = 0$, X is already diagonal.) This finally results in

$$\mathcal{M}_3\Big(\mathcal{M}_2\left(\mathcal{M}_1\mathcal{A}\mathcal{M}_1\right)\mathcal{M}_2\Big)\mathcal{M}_3 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mathrm{tr}(X) - \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{78}$$

and we have succeeded in diagonalizing $\mathcal{A}$ using F_4 as claimed.

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